Credit Spreads on Defaultable Sovereign Debt

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Abstract

In the 1970s Walter Wriston, then Citibank's CEO, famously said: "Countries do not go bankrupt". He was both right and wrong. He was right in the formal sense that no bankruptcy regime exists for sovereign borrowers. He was (very) wrong in that his statement intended to convey that, everything else being equal, sovereign loans were less risky than their corporate equivalents. This was proven wrong over and over again since the 1982 Latin American debt crisis, which pushed Citibank to the brink of insolvency. Indeed, precisely *because* of the lack of a formalized sovereign bankruptcy regime and the consequent lack of a credible threat of liquidation capable to wipe out the entirety of the borrower's wealth, all other things being equal, sovereign debt is more, not less risky than its corporate equivalent. Indeed, the lack of an "atomic option" in dealing with sovereign defaults results in giving sovereigns the ability to strategically default; i.e. to cease meeting their contractual obligations while they still have the ability to honor them and to subsequently engage in a restructuring which entails immediate cash-flow benefits (a relatively short period of non-payment followed by a longer one providing cash-flow relief with respect to the original contractual terms). The goal of this article is to develop a model which, given these idiosyncrasies allows for the valuation of sovereign debt instruments and their corresponding credit spreads.

1 Introduction

While the corporate corporate credit literature is rapidly advancing, the advances in the understanding of sovereign credits have lagged.

Sovereign debt cannot be valued by corporate debt valuation models because there is one major difference: a defaulting sovereign's asset cannot be liquidated in a bankruptcy proceeding (see Abadi, 2019 for a proposal of a possible sovereign bankruptcy mechanism). This implies that there are greater incentives for sovereigns than for corporates to strategically default; i.e., to pay less than the contractual amount even if there are enough resources to fulfill the contract in full.

This article, therefore, focuses on a country's willingness to honor its obligations rather than on its ability to do so – with such ability obviously acting as a constraint.. This framework of analysis is a major departure that used to calculate corporate credit risk. Hence, we develop our model by first modeling the sovereign's credit as if it were a corporate and then extending it to incorporate idiosyncratic features.

For the sovereign, the benefit of restructuring is that the existing stock of debt will be exchanged for new bonds with lower aggregate principal and coupon.

The adverse consequences to the sovereign will be slower country wealth growth resulting from direct trade sanctions (in the Bulow-Rogoff meaning) and reputational loss in the capital markets and elsewhere. Furthermore, the defaulting sovereign will expose itself to the threat of litigation from holdout creditors.

Our model builds upon corporate credit risk models by layering on the idiosyncratic features of sovereign debt. The starting point is the endogenous bankruptcy rule from Kim, Ramasawamy, and Sundaresan[1], where corporations default when the firm does not generate enough cash flow to service its debt.

We begin by modeling the dynamics of a country's wealth, including debt service. Following Bulow and Rogoff[2], we assume that terminal wealth maximization is the country's single utility function. This assumption is limiting, as it does not allow for the possibility that politicians' interests may differ from those of the citizenry.

We model the default sequence as a cessation of debt payments by the sovereign, followed by an exchange offer and a post default reduction in the rate of wealth growth due to sanctions, trade disruption, and reputational loss.

For analytic simplicity we assume a simple exchange of the defaulted bonds for new ones with the same maturity but the same proportional reduction in principal and coupon. For example, a sovereign bond with \$100 principal and 10% coupon will be exchanged into a bond with \$40 principal and 4% coupon, with the same maturity. Of course, apart from the oddity of a proportionally identical principal and coupon reduction, real-world sovereign bond exchanges almost always consist of exchanging shorter maturities for longer

[1] *Does default risk in coupons affect the valuation of corporate bonds?*, Kim, Ramaswamy, Sundaresan, 1993

[2] *A constant recontracting model of sovereign debt*, Bulow and Rogoff, 1989

ones. Our model deals with this reality by implicitly incorporating the NPV loss due to maturity extension into the principal and coupon reduction.

The sovereign's decision to default is made by comparing the expected terminal wealth under a continued debt service and a cost-of-default scenario. We define the *endogenous default boundary* as the level of sovereign wealth at which the country is indifferent between defaulting or not.

We further assume that the sovereign will always offer a deal such that the creditors are indifferent between accepting the exchange and litigating (see Abadi, 2019 for a more nuanced treatment of this dynamic), thus endogenizing the fraction of the original stock of debt the sovereign continues to honor post-default.

The result of the above assumptions is that we can derive a closed-form solution to the default-triggering wealth level, sovereign bond prices, and credit spreads. The endogenous repayment fraction, however, has only a numerical solution. This numerical analysis reveals a significant inverse relationship between sovereign credit spreads and litigation recoveries. As can be expected, the distance between a sovereign's wealth and the default boundary is also negatively correlated to the credit spread.

This article is organized as follows. Section 2 describes the model setup, while section 3 focuses on the calculation of the default boundary. Section 4 addresses the haircut, Section 5 offers the results of the numerical simulations of the endogenous default boundary and the resulting credit spreads. Section 6 concludes.

2 Model Specification

We consider a borrower with a single foreign currency-denominated bond outstanding. We assume that the country's wealth follows a Geometric Brownian Motion (GBM), and define it as all the assets either directly controlled by the sovereign's government or indirectly controlled by the sovereign's ability to seize within its territory assets susceptible to be converted into hard currency. The currency fluctuation effects are included in the state variable which determines country wealth and debt value. We postulate that the country's goal is to maximize its wealth as of the bond's maturity.

We further assume that the sovereign's bondholders are risk-neutral and identical (for a more nuanced treatment of investor behavior, see Abadi, 2019). Their only constraint is a non-negative expected IRR; i.e., they demand to at least get their principal back.

The sovereign has a choice between honoring the contract or defaulting. Under the default scenario the sovereign will cease servicing the bond and offer a new bond with lower principal and coupon in exchange. Bondholder, in turn, have a choice between accepting the exchange offer or litigating. The sovereign knows that, even when sovereign immunity is waived, sovereign litigation is expensive and the judgment-enforcement mechanisms are weak and will design its offer to be infinitesimally preferable to the litigation alternative.

In our model, the sovereign's cost of defaulting, including reputational ones, market access costs (both to trade and the capital markets), and output losses due to the resulting reduced capital inflows, are all subsumed in a single parameter which reduces the rate of growth of future sovereign wealth (i.e., in our model, the cost of defaulting is expressed in terms of lost growth opportunities, resulting in reduced future sovereign wealth).

By assumption, we model the relative change in sovereign wealth, R_t, as a GBM:

$$dR_t = (\mu - \delta)R_t dt + \sigma R_t dW_t \tag{1}$$

Where μ is the rate of sovereign wealth growth, δ is the total outflow (consumption plus debt service), and $\sigma R_t W_t$ is a diffusion term capturing the shocks to the economy.

The sovereign has a single bond outstanding with principal P, maturity T, and paying a coupon c. The outflow rate, σ, includes the sovereign's debt service payments, cP, as well as its consumption.[3]

Default occurs when sovereign wealth falls below the critical value, K_t, which we call the default boundary. If sovereign wealth decreases the costs of default also decrease, since they are assumed to be proportional to sovereign wealth. On they other hand, the benefits from default; i.e. the reduced debt service, remain constant since they are dependent only on the principal, P, and the coupon, c, and independent of sovereign wealth. Therefore, the country defaults when its sovereign wealth falls enough that the costs and benefits of default become equal; i.e. the default boundary is reached.

In this section we will first treat K_t as an exogenous, known constant (K_t will be endogenized in Section 3). If the sovereign defaults at time t, it offers a new bond in exchange which promises to pay only a fraction, α, of the original contractual obligation.[4] This implies that, from that time on, total debt service will be only αcP, with a principal repayment of αP at maturity. While here we treat α as an exogenous, known, constant, we will endogenize it in Section 4.

Because of the default, the country will suffer a reduction, $\lambda R_t dt$, in its currency inflows; i.e., the rate of growth in sovereign wealth will be reduced from the time of default. One interpretation of λ is that, as a consequence of the country's loss of reputation as a reliable trade partner, banks and investors will be reluctant to finance its trade after default.

Thus the post-default change in sovereign wealth becomes:

$$dR_t^D = (\mu - \delta - \lambda)R_t^D dt + \sigma R_t^D dW_t \tag{2}$$

[3]We assume that the sovereign will always consume an amount such that the sum of debt service and consumption will be equal to the outflow rate δ; i.e., we impose that total consumption is equal to $\delta - cP/R_t$, such that the total outflow is $\delta R_t dt = cPdt - \left(\delta - \frac{cP}{R_t} R_t dt\right)$.

[4]As stated in the introduction, sovereign debt restructurings almost always involve a maturity extension. In fact, the Argentine bond exchange currently being considered, involves *only* a maturity extension, with principal, P, and coupon, c, remaining unaffected. We note, however, that a maturity extension can be easily assimilated to a principal haircut and/or coupon reduction since the maturity extension reduces the bond's NPV. For example, Argentina's hypothetical offer to repay the principal at $T^{'} > T$, as originally promised, entails a loss for bondholders of $1 - e^{-r(T^{'} - T)}P$, since they will receive the principal $T^{'} - T$ years later than promised This is equivalent to a principal reduction of $\alpha = e^{-r(T^{'} - T)}$.

By defining $f(s; R_t; K)$ as the density of the first passage time, s, to the default boundary, K, we can calculate the value of a single bond with principal P and coupon c, as follows:

$$\begin{aligned} b(T; R_t) = &\int_0^T e^{-rs} cP[1 - F(s; R_t, K)]ds \\ &+ e^{-rT} P[1 - F(T; R_t, K)] \\ &+ \int_0^T \Big[\int_s^T e^{-ru} \alpha cP du + e^{-rT} \alpha P \Big] f(s; R_t, K) ds \end{aligned} \tag{3}$$

The first term is the expected value of the stream of discounted coupon payments, given that the sovereign is solvent. The probability of this event is the complement of the probability that the country's wealth hits the reorganization boundary, K, over the life of the bond. The second term is the expected discounted principal repayment, given that the sovereign remained solvent until maturity. The third term are the expected discounted coupon payments, αc, of the restructured debt after default, and the reduced expected discounted principal repayment, αP, in case of a default prior to maturity.

We can integrate by parts to simplify Eq. (3) to:

$$\begin{aligned} b(T; R_t, K) = &\frac{cP}{r} - e^{-rT} \frac{cP}{r} + e^{-rT} \frac{cP}{r} F(T; R_t, K) \\ &- \int_0^T e^{-rs} \frac{cP}{r} f(s; R_t, K) ds \\ &+ e^{-rT} P[1 - F(T; R_t, K)] \\ &+ \int_0^T \Big[\frac{\alpha cP}{r} (e^{-rs} - e^{-rT}) + e^{-rT} \alpha P \Big] f(s; R_t, K) ds \end{aligned}$$

which we can further simplify to find the value of the bond.

Proposition 1. *The value of a sovereign bond paying a coupon c p.a. and principal P at maturity T and which, after country wealth falls below a constant default-triggering sovereign wealth level K and which, after default, is exchanged into a bond paying a coupon αc and a final repayment αP at maturity, is, at time t, given by:*

$$\begin{aligned} b(T; R_t, K) = &\frac{cP}{r} + e^{-rT} \Big(P - \frac{cP}{r} \Big) [1 - F(T; R_t, K)] \\ &+ \alpha e^{-rT} \Big(P - \frac{cP}{r} \Big) F(T; R_t, K) \\ &+ (\alpha - 1) \frac{cP}{r} \int_0^T e^{-rs} f(s; R_t, K) ds \end{aligned} \tag{4}$$

A solution for $F(T;R_t,K)$ can be found in Harrison[5], and a solution for $\int_0^T e^{-rs}f(s;R_t,K)ds$ in Rubinstein and Krieger[6]. Both of these solutions assume that K is constant.

$$F(T;R_t,K) = n(h_{1,T}) + \Big(\frac{R_t}{K}\Big)^{-2a} N(h_{2,T}) \tag{5}$$

$$\int_0^T e^{-rs} f(s-t;R_t,K)ds = \big(\frac{R_t}{K}\Big)^{-a+g} N(q_{1,T} + \Big(\frac{R_t}{K}\Big)^{-a-g} N(q_{2,T}) \tag{6}$$

with

$$a = \frac{\mu - \delta - (\sigma^2)}{\sigma^2};$$

$$b = ln(\frac{R_t}{K});$$

$$g = \frac{\sqrt{(a\sigma^2)^2 + 2r\sigma^2}}{\sigma^2};$$

$$h_{1,T} = \frac{-b - a\sigma^2 T}{\sigma\sqrt{T}};$$

$$q_{1,T} = \frac{-b - g\sigma^2 T}{\sigma\sqrt{T}};$$

$$h_{2,T} = \frac{-b + a\sigma^2 T}{\sigma\sqrt{T}};$$

$$q_{2,T} = \frac{-b + g\sigma^2 T}{\sigma\sqrt{T}}.$$

3 Endogenous Default Boundary

So far we have treated the default boundary, K, as exogenous and constant; i.e. we didn't allow for the possibility of a strategic default. To allow for this possibility, we keep the assumption the country's goal is to maximize sovereign wealth net of consumption and debt repayment at the bond's maturity, which will depend on sovereign decisions: at each time s, prior to the maturity T the sovereign decides whether to default or not by comparing the sovereign wealth resulting from each strategy.

We define s as the time when the relative change in sovereign wealth, R_t, hits the default boundary, K_t, and $V_t(s)$ as the sovereign wealth, net of consumption but including debt payments. The dynamics of $V_t(s)$ are a function of the sovereign's decision to default. Prior to default, $V_t(s)$ equals the sovereign wealth growth rate, μ, minus the contractual

[5] *Brownian motion and stochastic flow systems.* Krieger Publishing Company, 1990

[6] *Breaking down the barriers.* Risk Magazine, 4:28-35, 1991

coupon payment, cP. After default (and exchange), the growth $V_t(s)$ slows to $\mu - \lambda$, due to the costs of default, but only minus αcP, the post-exchange coupon.

We assume that the sovereign pays off the principal αP at maturity of the exchanged bond and does not borrow again. However, we assume that, since the sovereign defaulted, its sovereign wealth growth rate will continue to suffer from the damage, λ in perpetuity.

If the sovereign defaults at time s, we define $V_t(s)$[7] by:

$$dV_t(s) = \begin{cases} (\mu V_t^N - cP)dt + \sigma V_t^N dW_t & \text{if } t < s \text{ and } t < T \\ \big((\mu - \lambda)V_t^D - \alpha cP\big)dt + \sigma V_t^D dW_t & \text{if } t \geq s \text{ and } t < T \\ \mu V_t^N dt + \sigma V_t^N dW_t & \text{if } T < s \text{ and } t > T \\ (\mu - \lambda)V_t^D dt + \sigma V_t^D dW_t & \text{if } T > s \text{ and } t > T \end{cases} \quad (7)$$

(7)

Since the sovereign repays the principal at T, we can define the sovereign wealth net of consumption and principal repayment as:

$$V_{T^*} = \begin{cases} V_t^N(s) - P & \text{if } T < s \\ V_t^D(s) - \alpha P & \text{if } T > s \end{cases} \quad (8)$$

(8)

where T^* is the time just after principal repayment.

The value of the no-default strategy, $U^N(s)$ then is:

$$U^N(s) = E_s[V_{T^*}(s)|s > T] = E_s[V_t^N|s > T] - P \quad (9)$$

Instead, if the sovereign decided to default at s, it would pay a lower coupon, αc, and principal. αP, but its sovereign wealth would grow at the lower rate given by Eq.(2) going forward. Since the slower growth rate continues after the bond's repayment, the sovereign must incorporate this perpetual slower growth in its decision.

Since we assumed that the sovereign does not re-borrow after the bond's maturity, the dynamics of the sovereign wealth net of consumption, become $dV_t^D(s) = (\mu - \lambda)V_T^D(s)dt +_t^D$

[7]For clarity, we use the nomenclature $V_t^D(s)$ and $V_t^N(s)$ for the sovereign wealth with default and no default, respectively

$(s)dW_t$, with $t > T$ and $s < T$. Had the country not defaulted at s, its sovereign wealth would be given by $dV_tN(s) = \mu V_t^N dW_t$, with $t > T$. Therefore, the log-difference between the two sovereign wealth dynamics is $d\ln V_t^D - d\ln V_t^N = -\lambda dt$. Knowing the difference allows us to calculate (under risk-neutrality) the expected sovereign wealth growth loss discounted to T as:

$$
\begin{aligned}
\Delta V &= E_s\Big[V_{T^*}^D \int_T^\infty \lambda e^{-r(u-T)}\big(V_u^D(s) - V_u^N(s)\big)du\Big|V_s, s < T\Big] \\
&= \lambda e^{rT} E_s\Big[V_{T^*}^D \int_T^\infty exp\big\{-ru(\ln V_t^D - d\ln V_t^N)\big\}du\Big|V_s, s < T\Big] \\
&= \lambda E_s V_{T^*}^D(s)|V_s, s < T] \int_0^\infty exp\{-(r+\lambda)(u-T)\}du \\
&= \frac{\lambda}{\lambda + r} E_s[V_{T^*}^D(s)|V_s, s < T] \qquad (10)
\end{aligned}
$$

where $E_s[V_{T^*}^D(s)|s < T] = E_s[V_t^D(s)|s < T] - \alpha P$ is the expected sovereign wealth just after repayment of the principal. We can now calculate $U^D(s)$, the expected value of the default strategy discounted to time T. It is going to be simply the expected sovereign wealth at T (net of consumption), given that the sovereign defaulted at s, less the principal repayment.[8]

$$
\begin{aligned}
U^D(s) &= E_s[V_{T^*}^D|s < T] - \frac{\lambda}{\lambda + r} E_s[V_{T^*}^D(s)|s < T] \\
&= \Big(\frac{\lambda}{\lambda + r}\Big)(E_s[V_t^D|s < T] - \alpha P) \qquad (11) \\
&= \frac{r}{\lambda + r}(E_s[V_t^D|s < T] - \alpha P)
\end{aligned}
$$

It is strategically optimal to default at time s whenever the expected value of defaulting, $U^D(s)$, is greater than the expected value of continuing to honor the sovereign's obligation in full, $U^N(s)$, conditional on the information available at time s. At the default boundary, $K(s, T, \alpha)$, the sovereign is indifferent between defaulting and not defaulting. We can, therefore, calculate $K(s, T, \alpha)$ as the value V_s^* which satisfies:

$$
\begin{aligned}
& U^N(s) = U^D(s) \\
\Rightarrow \quad & E_s[V_t^N|s > T] - P = r\lambda + r(E_s[V_t^D|s < T] - \alpha P) \qquad (12)
\end{aligned}
$$

To solve for $K(s, T, \alpha)$ we need to calculate the conditional expectation at time s of the sovereign wealth (net of consumption) at maturity of the bond:

[8] Naturally, since the sovereign defaulted, the calculation incorporates the cost of default, the reduced debt payments, and the present value of the post-maturity perpetual growth loss

- With default and time s:

$$E_s[V_t^D|s<T] = V_s e^{(\mu-\lambda)(T-s)} + \frac{\alpha c P\big(1-e^{(\mu-\lambda)(T-s)}\big)}{\mu-\lambda} \tag{13}$$

- With no default:

$$E_s[V_t^N|s>T] = V_s e^{\mu(T-s)} + \frac{cP\big(1-e^{\mu(T-s)}\big)}{\mu} \tag{14}$$

We can now solve for Eq.(12)

Proposition 2. *The country will default when its sovereign wealth hits the endogenous default boundary, $K(s,T,\alpha)$, given by:*

$$K(s,T,\alpha) = P\frac{1+\frac{c}{\mu}\big(e^{\mu(T-s)}-1\big) - \frac{r}{r+\lambda}\alpha\big(1+\frac{c}{\mu-\lambda}\big(e^{(\mu-\lambda)(T-s)}-1\big)\big)}{e^{\mu(T-s)} - \frac{r}{r+\lambda}e^{(\mu-\lambda)(T-s)}} \tag{15}$$

In order to fully understand the thought process underlying a strategic sovereign default, we need to realize that, while the potential benefits of default are independent of the sovereign's wealth, the costs of default are not. Therefore, should sovereign wealth drop below the endogenous default boundary, the costs of default will be less than its benefits.

3.1 Valuation of the Bond with Time-Dependent Endogenous Default Boundary $K(s,T,\alpha)$

Since the endogenized default boundary is *time-dependent*, and Eqs. (5) and (6) require a *constant boundary*, while $K(s,T,\alpha)$ is curved, we cannot calculate the first-passage densities required to value the bond in closed-form.

Instead, we can use the Durbin approximation[9] to numerically compute the first-passage time densities required to transform the process R_t. Durbin shows that the first-passage Brownian motion, W_t, to a curved boundary $a(s)$ at $t=s$ can be well approximated by:

$$\begin{aligned} f(s,a(s)) \approx & \Big[\frac{a(s)}{s} - a^{'}(s)\Big]\phi(s) \\ & - \int_0^s \Big[\frac{a(u)}{u} - a^{'}(u)\Big]\Big[\frac{a(s)-a(u)}{s-u} - a^{'}(s)\Big]\phi(u,s)du \end{aligned} \tag{16}$$

where $a^{'}(s)$ is the derivative of $a(s)$, $\phi(s)$ is the density of the Brownian motion W_t at time s, evaluated at $a(s)$, and $\phi(u,s)$ is the joint density of W_t at times u and s, evaluated at $a(u)$ and $a(s)$.

[9] The First Passage Density of the Brownian Motion Process to a Curved Boundary, *Journal of Applied Probability*, 1992

Default occurs in our model when sovereign wealth, R_t drops below the default boundary, K_t. Therefore, we can transform the problem into the first hitting time of a standard Brownian motion to a transformed boundary, $A(s)$, defined in the interval $[0, T]$, as:

$$A(s) = \frac{\ln\left(\frac{K(s,T,\alpha)}{R_t}\right) - \left(\mu - \delta - \frac{1}{2}\sigma^2\right)s}{\sigma} \tag{17}$$

In order to evaluate Eq. (16) we need to first calculate the derivative of the boundary as:

$$\begin{aligned} a^{'}(s) &= \frac{\partial A(s)}{\partial s} = \frac{1}{\sigma}\frac{\partial \ln K(s,T,\alpha)}{\partial s} - \frac{\mu - \delta - \frac{1}{2}\sigma^2}{\sigma} \\ &= \left[\frac{\frac{r}{r+\lambda}\alpha c e^{(\mu-\lambda)(T-s)} - c e^{\mu(T-s)}}{K(s,T,\alpha)}P - \psi(s)\right]\sigma^{-1}\Psi(s)^{-1} \\ &- \frac{\mu - \delta - \frac{1}{2}\sigma^2}{\sigma} \end{aligned} \tag{18}$$

with

$$\Psi(s) = e^{\mu(T-s)} - \frac{r}{r+\lambda}e^{(\mu-\lambda)(T-s)}$$

and

$$\psi(s) = \frac{\partial \Psi(s)}{\partial s} = \frac{r(\mu-\lambda)}{r+\lambda}e^{(\mu-\lambda)(T-s)} - \mu e^{\mu(T-s)}$$

Now we need the density of the Brownian motion, W_t, at time s and the joint density at times u and s, evaluated at $A(s)$ and $A(u)$, respectively:

$$\phi(s) = \frac{1}{\sqrt{2\pi(s-t)}}\exp\left\{-\frac{1}{2}\frac{(A(s)-A(t))^2}{s-t}\right\} \tag{19}$$

$$\phi(u,s) = \phi(s)\frac{1}{\sqrt{2\pi(u-s)}}\exp\left\{-\frac{1}{2}\frac{(A(u)-A(s))^2}{u-s}\right\} \tag{20}$$

We can now easily calculate the cumulative default probability $F(T-t; A(s)) = \int_t^T f(s,t,A(s))ds$ and the expression $\int_t^T e^{-rs} f(s,t,A(s))ds$ with $f(s,t,A(s)$ as defined by Eq. (16) and $A(s)$ as defined by Eq. (17). We plug these results into Eq. (4) and we get the value of the bond:

Proposition 3. *Assuming that a sovereign's goal is to maximize sovereign wealth, the value of a sovereign bond with coupon c, principal P and maturity T which defaults after sovereign wealth falls below an endogenous default-triggering sovereign wealth level $K(s,T,\alpha)$ is approximated by:*

$$\begin{aligned} b(t,T;R_t,\alpha,A(s)) = &\frac{cP}{r} + e^{-r(T-t)\left(P-\frac{cP}{r}\right)[1-\int_t^T f(s,t,A(s))ds]} \\ &+ \alpha e^{-r(T-t)\left(P-\frac{cp}{r}\right)\int_t^T f(s,t,A(s))ds} \\ &+ (\alpha-1)\frac{cP}{r}e^{rt}\int_t^T e^{-rs} f(s,t,A(s))ds. \end{aligned} \tag{21}$$

with

$$f(s,t,A(s)) \approx \Big[\frac{A(s)}{s} - A^{'}(s)\Big]\phi(s) - \int_t^T \Big[\frac{A(u)}{u} - A^{'}(u)\Big]\Big[\frac{A(s)-A(u)}{s-u} - A^{'}(s)\Big]\phi(u,s)du.$$

and

$$A(s) = \frac{ln\big(\frac{K(s,T,\alpha)}{R_t}\big) - \big(\mu-\delta-\frac{1}{2}\sigma^2\big)(s-t)}{\sigma}$$

$$K(s,T,\alpha) = P\frac{1+\frac{c}{\mu}\big(e^{\mu(T-s)} - \frac{r}{r+\lambda}\alpha\big(1+\frac{c}{\mu-\lambda}\big(e^{(\mu-\lambda)(T-s)}-1\big)\big)}{e^{\mu(T-s)} - \frac{r}{r+\lambda}e^{(\mu-\lambda)(T-s)}}$$

$$\phi(s) = \frac{1}{\sqrt{2\pi(s-t)}}exp\Big\{-\frac{1}{2}\frac{(A(s)-A(t))^2}{s-t}\Big\}$$

$$\phi(u,s) = \phi(s)\frac{1}{\sqrt{2\pi(u-s)}}exp\Big\{-\frac{1}{2}\frac{(A(u)-A(s))^2}{u-s}\Big\}$$

$$A^{'}(s) = \Bigg[\frac{\frac{r}{r+\lambda}\alpha c e^{(\mu-\lambda)(T-s)} - c e^{\mu(T-s)}}{K(s,T,\alpha)}P - \psi(s)\Bigg]\sigma^{-1}\Psi(s)^{-1} - \frac{\mu-\delta-\frac{1}{2}\sigma^2}{\sigma}$$

$$\Psi(s) = e^{\mu(T-s)} - \frac{r}{r+\lambda}e^{(\mu-\lambda)(T-s)}$$

$$\psi(s) = \frac{\partial\Psi(s)}{\partial s} = \frac{r(\mu-\lambda)}{r+\lambda}e^{(\mu-\lambda)(T-s)} - \mu e^{\mu(T-s)}$$

3.2 Special Case: Perpetual Bond

It we let the sovereign bond's maturity go to infinity, the default boundary becomes constant again:

$$K^{infty} = \lim_{T\to\infty} K_s = \frac{cP}{\mu} \tag{22}$$

If we then apply the Proposition 1 valuation formula, we get the value of the perpetual sovereign bond:

$$b^{\infty}(t;T_t,K^{\infty}) = \frac{cP}{r} + (\alpha-1)\frac{cP}{r}e^{rt}\big(\frac{R_t}{K^{\infty}}\big)^{-a-g} \tag{23}$$

with

$$a = \frac{\mu-\delta-(\frac{\sigma^2}{2})}{\sigma^2}$$

$$g = \frac{\sqrt{(a\sigma^2)^2 + 2r\sigma^2}}{\sigma^2}$$

4 Endogenizing the Repayment Fraction α

When the sovereign reaches the default boundary $K(s, T, \alpha)$, it offers its creditors to exchange the current bond for a new bond with the same maturity but paying only a fraction,α, of the original coupon, c and principal, P. Assuming that the exchange takes place at time s, the market value of the new bond is $b(s; T; R_s, \alpha c, \alpha P, A(s))$. If the creditors reject the exchange offer they litigate and recover a fraction $(1-\gamma)P$[10] of the principal owed. We further assume that the sovereign will seize the haircut to be such that, given the expected litigation costs, creditors will be about indifferent accepting the offer or litigating for full repayment.

Proposition 4. *The endogenous repayment fraction, α is the alpha* which satisfies the equation:*

$$b(t, T; R_s, \alpha^*, A(s)) = (1 - \gamma)P \tag{24}$$

but since $R_s = K(s, T, \alpha^*)$

$$b(t, T; K(s, T, \alpha^*), \alpha^*, A(s)) = (1 - \gamma)P \tag{25}$$

Proposition 3 provided a closed-form solution for the value of the defaulted sovereign bond. By using that value to solve numerically for the value of α^* which equates the value of the exchange offer to the expected value of recoveries from litigation. By plugging the value of the numerically-derived α^* into the proposition 3 equation gives us the current value of the sovereign bond:

$$b(t, T; R_t, \alpha^*, A(s)).$$

5 Sovereign Debt Credit Spread Sensitivities

The formula for sovereign credit spreads does not have a closed-form solution. However, we can numerically solve for for the endogenous repayment fraction we calculated in Eq. (25) and plug the endogenous default boundary we derived in Eq. (15) into the bond valuation formulas [Eqs. (4), (5), and (6)] to derive the sovereign bond price and, hence, its spread over treasurys:

$$S_{t,T} = \frac{cP}{b(t, T; R_t, \alpha^*, A(s))} - \frac{cP}{(1 - e^{-rT})\frac{cP}{r} + Pe^{-rT}} \tag{26}$$

Since not all of S_{t_T}'s derivatives have a closed-form solution, we will rely on the intuitions provided by the formula to ascertain the effect of each relevant variable on the sovereign credit spread $S_{t,T}$. We will, instead, infer the effects of each variable on the sovereign credit spread through the intuitions the equation reveals.

[10]This assumption is different from Gibson and Sundaresan's (1999), who assume that the litigants' recovery value is a function of sovereign wealth and future exports. We, instead assume that the litigants cannot recover more than the principal minus litigation costs; i.e., litigation recovery is independent from sovereign wealth.

5.1 Sensitivity to Litigation Costs γ

The litigation costs, γ determine the recovery from litigation. The higher the litigation costs, the higher the haircut $(1-\alpha)$ the sovereign can offer in the exchange and still leave the creditors marginally more willing to accept than to reject. In turn, the higher the haircut, the more attractive default becomes from the sovereign's viewpoint; i.e. the sooner the sovereign will choose to default. Therefore, sovereign credit spreads rise as litigation costs rise.

5.2 Sensitivity to Initial Sovereign Wealth R_t

Obviously, the lower the initial sovereign wealth, R_t, the higher the probability that the country will hit the default boundary, K_t and hence, all other things being equal, sovereign credit spreads increase as initial sovereign wealth decreases.

5.3 Sensitivity to Cost of Default Rate λ

A higher the sovereign wealth growth rate loss due to default, λ will act as a deterrent to default. Therefore, the higher the λ the lower the sovereign credit spread.

5.4 Sensitivity to Sovereign Wealth Volatility σ

Although numerical analysis shows the relationship not to be monotonous. Initially, sovereign credit spreads increase with sovereign wealth volatility due to the bigger probability that sovereign wealth level will drop below the endogenous default boundary, K_t. However, as the repayment fraction, α, also increases with sovereign wealth volatility, the second phase of the low-volatility interval will be characterized by sovereign credit spreads falling with sovereign wealth volatility. Finally, beginning with a ”steady state” sovereign wealth volatility (e.g., 30%), the correlation between sovereign credit spreads and sovereign wealth volatility becomes positive again.

5.5 Sensitivity to Coupon Rate c

High-coupon debt increase a sovereign's incentive to default, since it leads to higher savings as a result. As a result, the endogenous default boundary, K_t, increases as the coupon, c, increases which, in turn, compels the country to offer better exchange conditions in order to ensure participation in the exchange. Therefore, since the country will default at higher levels of sovereign wealth, its default probability is higher and hence, the sovereign credit spread increase with the coupon.

5.6 Sensitivity to Maturity T

Every practitioner is familiar with the phenomenon that, as default looks inevitable, sovereign bonds shift from trading on a yield basis to a price basis. This leads to short maturities trading at much higher spreads than long ones. In more formal terms, while the costs of default are independent of the maturity, the benefits from default depend on the time to maturity through two distinct channels:

- Discounting: Later cash flows have higher discount factors; they will be smaller in NPV terms.
- Lifetime: Total future coupons on longer maturity bonds are higher than on short maturity bonds

From the viewpoint of the sovereign, it is attractive to default on short maturity bonds because this behavior maximizes principal savings. Therefore, the endogenous default boundary is high for short maturity bonds. As the time to maturity increases, however, these principal savings become smaller in NPV terms, rendering default less attractive, as the endogenous default boundary drops with as time to maturity increases. This, the sovereign credit spread curve is hump-shaped: very short maturity sovereign bonds will trade at relatively tight spreads because of the expectation that they will be repaid before default is declared. Beyond this very narrow interval of quasi-immediate maturities, sovereign credit spreads increase dramatically along with the default probability. Sovereign credit spreads will start decreasing again for bonds with longer contractual maturities.

6 Conclusion

This article presented an analysis of a sovereign's decision to strategically default.

In addition, our highly-simplified model allowed us, through numerical analysis, to isolate and rank the most important determinants of sovereign bond credit spreads, as follows:

1. expected litigation recovery;
2. distance of the sovereign's wealth to the endogenous default boundary; 0agnitude of the coupon on debt outstanding prior to default; and
3. volatility of sovereign wealth.

Our model explains other stylized facts of sovereign defaults, such as the fact that sovereigns tend to default when faced with a cluster of near maturities.

However, for the model to be useful toward sovereign sovereign debt pricing/valuation, the model needs to be expanded to include three parameters not included in our simplified model:

1. the actual case of default for the sovereign, since *creditors are not negotiation-proof*. Indeed a small payment made to creditors after default could reduce the sovereign's growth drag due to default;

2. sovereigns tend not to default in their total debt, especially in the case of strategic defaults. An extension of our model would be required to analyze the more general case where, for strategic reasons, a sovereign may decide to remain in compliance with some sovereign debt contracts; and

3. creditors are not homogeneous since they don't all share the same litigation costs. A more nuance explanation of the phenomenon whereby creditors are segmented by their propensity co litigation can be found in Abadi, 2019.