Fairness in Talmudic "Bankruptcy Law"

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Abstract

The bankruptcy problem is as old as money itself. The main challenge to analyzing ancient solutions to the bankruptcy problem is that these solutions are often presented not as a general algorithm, but rather as a series of examples. Whenever those examples are few in number, it is often difficult to derive the underlying algorithm. This was the case, for example, with the Talmudic solution to the bankruptcy problem. Only in the mid-1980s did Nobelist Robert Aumann succeed in cracking the algorithm underlying the Talmudic "Bankruptcy Code". Aumann used game theory techniques to crack the code but did not delve into the question of whether such code was fair. In this paper we attempt to answer that question.

1 Introduction

A person or company goes bankrupt when they can no longer satisfy their obligations in full. Available funds (the estate) are then distributed among the claimants. Since there is not enough money to satisfy everybody in full, at least some claimants will recover less than they're owed. How much less? Is there a fair way to divide the estate among the claimants?

1.1 The Talmudic Solution to the Bankruptcy Problem

The bankruptcy problem is as old as money itself and solutions to the problem have been proposed for about just as long. One ancient solution is posited in the Babylonian Talmud (Ketubot 93a, Bava Metzia 2a, and Yevamot 38a).

Like many ancient texts dealing with numbers, the Talmud does not offer an explicit algorithm. Instead, it puts forward four examples illustrating how an estate (E) should be divided. In the first three examples, three parties are owed the following amounts:

- the first claimant is owed $d_1 = 100$,
- the second claimant is owed $d_2 = 200$, and

- the third claimant is owed $d_3 = 300$

$$d_1 = 100, d_2 = 200, d_3 = 300$$

For three different amounts of E, the text prescribes the amounts $e_1, e_2, \text{and} e_3$ that each claimant will recover:

	$d_1 = 100$	$d_2 = 200$	$d_3 = 300$
E	e_1	e_2	e_3
100	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
200	50	75	75
300	50	100	150

The fourth example was formulated sightly differently – as the question of dividing a disputed garment. In bankruptcy terms, there was one garment and one claimant claimed to be entitled to half of it, while the second claimed to be entitled to all of it. Following the nomenclature above:

$$d_1 = 50, d_2 = 100$$

The estate E and the split prescribed by the Talmud are as follows:

	$d_1 = 50$	$d_2 = 100$
E	e_1	e_2
100	25	75

1.2 Aumann's Algorithm

The problem posed by these four examples intrigued Professor Robert Aumann, who went on to become the Nobel Prize winner in economics (2005). In his 1985 seminal paper, *Game Theoretic Analysis of a Bankruptcy Problem from the Talmud*, Professor Aumann put forward a general algorithm that fit the data for all four examples.

1.2.1 The case of two claimants

In order to explain Aumann's algorithm, we need to first start with the case of two claimants. Without losing generality, let us assume that the first claimant has a smaller claim (i.e., $d_1 \leq d_2$). For his analysis, Aumann distinguished three cases:

- When the estate's size E is small, i.e. smaller than d_1, then E is distributed equally between claimants, so that each gets:

$$e_1 = e_2 = \frac{E}{2}$$

- When the estate E is between d_1 and d_2 ($d_1 \leq E \leq d_2$), the the first claimant recovers $e_1 = d1/2$ and the second claimant recovers the remaining amount $e_2 = E - e_1$. This policy continues until we reach the amount $E = d_2$, at which time the first claimant receives $e_1 = d_1/2$ and the second claimant receives the remaining $e_2 = d_2 - d_1/2$. Thus, both claimants lose the same amount of money: $d_1 - e_1 = d_2 - e_2 = d_1/2$.

- Finally, when the estate's size E is larger than d_2 (but smaller than the total amount owed $d_1 + d_2$, the estate is divided in a way such that the losses remain equal, i.e. $d_1 - e_1 = d_2 - e_2$ and $e_1 + e_2 = E$. From these two conditions Aumann derived the corresponding recoveries:

$$e_1 = \frac{E + d_1 - d_2}{2}, e_2 = \frac{E - d_1 + d_2}{2}$$

1.2.2 The general case

Aumann then goes on to explain that the allocation among three (or more) claimants is such that for every two claimants the amounts recovered are allocated according to the above algorithm. This can be easily verified if we select, for each pair (i, j) only the estate $E_{ij} = e_i + e_j$ allocated to claimants from this pair. As a result, for the pairs $(1, 2), (2, 3)$ and $(1, 3)$, we get the following tables:

	$d_1 = 100$	$d_2 = 200$
E_{12}	e_1	e_2
$66\frac{2}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
125	50	75
150	50	100

	$d_1 = 200$	$d_2 = 300$
E_{23}	e_1	e_2
$66\frac{2}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
150	75	75
250	100	150

	$d_1 = 100$	$d_2 = 300$
E_{13}	e_1	e_2
100	$66\frac{2}{3}$	$33\frac{1}{3}$
125	50	75
200	50	150

1.3 But Is It Fair?

Thus Aumann reconstructed the sages' algorithm and we now know *what* the rabbis proposed. However it is still not clear *why* this solution to the bankruptcy problem was proposed.

There appears to be a hint of arbitrariness in the rabbis' solution. To be more precise, both the ideas of dividing the estate equally and dividing the losses equally make sense. But how do we combine both approaches? And why in the region between $E = min(d_1, d_2)$ and $E = max(d_1, d_2)$ the claimant with the smaller claim always recovers half while the other recovers more and more? How does that fit with the concept that the Talmud's proposed allocations are fair?

We'll explore that proposition in the next sections.

2 Problem Setup

As proposed by another Nobelist, John Nash (*Two-Person Cooperative Games*, 1953), in the case of two claimants, fairness must be defined with respect to each claimant's status quo ante $(\tilde{e_1}, \tilde{e_2})$ and any change from that status quo must be divided equally (i.e. $e_1 - \tilde{e_1} = e_2 - \tilde{e_2}$).

2.1 Two Claimant Case

Let's consider that we have two claimants. The first claimant is owed $d_1 = 100$ and the second $d_2 = 200$ and that there is an estate $E_{12} = 125$.

For the first claimant, the best case is that he recovers all the money he's owed, i.e. $\bar{e_1} = 100$ and the worst case (if the entire estate is adjudicated to the second claimant) is that he gets nothing, i.e. $\underline{e_1} = 0$. Therefore, the status quo for the first claimant lies somewhere within the interval:

$$[\underline{e_1}, \bar{e_1}] = [0, 100]$$

Similarly, for the second claimant, the best case is a full recovery, i.e. $\bar{e_2} = 125$ and the worst case (when the first claimant is fully satisfied) is $\underline{e_2} = 125 - 100 = 25$. Therefore, the status quo interval for the second claimant lies somewhere within the interval:

$$[\underline{e_2}, \bar{e_2}] = [25, 125]$$

2.2 General Case

Let us consider, without losing generality, that the first claimant is owed less than the second, i.e. $d_1 \le d_2$ and further consider the following three cases:

- the estate does not exceed the first claimant's claim, i.e. $E_{12} \le d_1$,
- the estate lies between the first and second claimant's amount, i.e. $d_1 \le E_{12} \le d_2$, and
- the estate exceeds the second claimant's claim, i.e. $d_2 \le E_{12} \le d_1 + d_2$.

We consider each of the above cases one at a time.

2.2.1 The size of the estate does not cover the largest claim

In this case ($E_{12} \le d_1 \le d_2$), the best case scenario for the first claimant is to receive the entire estate ($\bar{e_1} = E_{12}$) and the worst case is when all the money goes to the second claimant and the first claimant gets nothing ($\underline{e_1} = 0$). Therefore, for the first claimant the recovery range is $[\underline{e_1}, \bar{e_1}] = [0, E_{12}]$.

For the second claimant, the best case is to receive the entire estate ($\bar{e_2} = E_{12}$ and the worst case is to recover nothing ($\underline{e_2} = 0$). Therefore, for the second claimant the recovery range is $[\underline{e_2}, \bar{e_2}] = [0, E_{12}]$.

2.2.2 The size of the estate lies between the smallest and the largest claims

In this case ($d_1 \le E_{12} \le d_2$), the best case for the first claimant is to recover in full ($\bar{e_1} = d_1$) and the worst case is $\underline{e_1} = 0$. Therefore, the first claimant's recovery range is $[\underline{e_1}, \bar{e_1}] = [0, d_1]$.

For the second claimant, the best case is to recover the entire estate ($\bar{e_2} = E_{12}$) and the worst case is when the first claimant recovers in full and the second gets the remainder $\underline{e_2} = E_{12} - d_1$. Therefore, the recovery range for the second claimant is $[\underline{e_2}, \bar{e_2}] = [E_{12} - d_1, E_{12}]$.

2.2.3 The size of the estate exceeds both claims

In this case ($d_1 \le d_2 \le E_{12}$), the best case for the first claimant is to recover in full ($\bar{e_1} = d_1$) and the worst case is when the second claimant recovers in full and the first gets only the remainder $\underline{e_1} = E_{12} - d_1$. Therefore, for the first claimant the recovery range is $[\underline{e_1}, \bar{e_1}] = [E_{12} - d_2, d_1]$.

For the second claimant, the best case is to recover in full ($\bar{e_2} = d_2$) and the worst case is when the first claimant recovers in full and the second gets only what remains of the estate $\underline{e_2} = E_{12} - d_1$. Therefore, the recovery range for the second claimant is $[\underline{e_2}, \bar{e_2}] = [E_{12} - d_1, d_2]$.

2.3 Selection Within Intervals

In all three cases above, the recovery for each claimant is expressed as an *interval.* Therefore, we need to select a status quo point that corresponds to this interval uncertainty. The challenge of modeling the fair cost $\tilde{e}$ for the uncertainty of interval $[\underline{e}, \bar{e}]$ has been solved by yet another Nobelist, Leo Hurwicz (*Optimality Criteria for Decision Making Under Ignorance*, 1951) as follows:

$$\tilde{e} = \alpha\bar{e} + (1 - \alpha)\underline{e}$$

where the coefficient $\alpha \in [0, 1]$ quantifies the decision-maker's degree of optimism:

- the value $\alpha = 1$ describes a perfect optimist, a decision-maker who considers only the best-case scenario;
- the value $\alpha = 0$ describes a perfect pessimist, a decision-maker who considers only the worst-case scenario; and
- values of α strictly between 0 and 1 describe a realistic decision-maker who considers both the best and worst-case scenarios.

Let us consider each of the above scenarios as a status quo under the rule that a distribution is fair if the differences between the recoveries e_i and the status quo are equal: $e_1 - \tilde{e_1} = e_2 - \tilde{e_2}$.

3 We Get the Talmudic Solution Independently of the Degree of Optimism

We will now show that we get exactly the Talmudic solution for any value of α. To do that, we revisit the three cases in 2.2:

- the estate does not exceed the first claimant's claim, i.e. $E_{12} \leq d_1$,
- the estate lies between the first and second claimant's amount, i.e. $d_1 \leq E_{12} \leq d_2$, and
- the estate exceeds the second claimant's claim, i.e. $d_2 \leq E_{12} \leq d_1 + d_2$.

3.1 The size of the estate does not cover the largest claim

In this case,

$$\tilde{e_1} = \alpha \cdot \bar{e_1} + (1 - \alpha) \cdot \underline{e_1} = \alpha \cdot E_{12} + (1 - \alpha) \cdot 0 = \alpha \cdot E_{12}$$

and, similarly:

$$\tilde{e_2} = \alpha \cdot \bar{e_2} + (1-\alpha) \cdot \underline{e_2} = \alpha \cdot E_{12} + (1-\alpha) \cdot 0 = \alpha \cdot E_{12}$$

Therefore, the fairness condition $e_1 - \tilde{e_1} = e_2 - \tilde{e_2}$ becomes $e_1 - \alpha \cdot E_{12} = e_2 - \alpha \cdot E_{12}$, i.e. $e_1 = e_2$, which is the Talmudic distribution.

3.2 The size of the estate lies between the smallest and the largest claims

In this case,

$$\tilde{e_1} = \alpha \cdot \bar{e_1} + (1-\alpha) \cdot \underline{e_1} = \alpha \cdot d_1 + (1-\alpha) \cdot 0 = \alpha \cdot d_1$$

and

$$\tilde{e_2} = \alpha \cdot \bar{e_2} + (1-\alpha) \cdot \underline{e_2} = \alpha \cdot E_{12} + (1-\alpha) \cdot (E_{12} - d_1) = E_{12} - (1-\alpha) \cdot d_1$$

Therefore, the fairness condition $e_1 - \tilde{e_1} = e_2 - \tilde{e_2}$ becomes:

$$e_1 - \alpha \cdot d_1 = e_2 - E_{12} + (1-\alpha) \cdot d_1 = e_2 - E_{12} + d_1 - \alpha \cdot d_1$$

which simplifies to:

$$e_1 = \frac{d_1}{2} \text{ and } e_2 + E_{12} = \frac{d_1}{2}$$

Which, again, is the Talmudic distribution.

3.3 The size of the estate exceeds both claims

In this case,

$$\tilde{e_1} = \alpha \cdot \bar{e_1} + (1-\alpha) \cdot \underline{e_1} = \alpha \cdot d_1 + (1-\alpha) \cdot (E_{12} - d_2) = \alpha \cdot d_1 + (1-\alpha) \cdot (E_{12} - (1-\alpha) \cdot d_2$$

and

$$\tilde{e_2} = \alpha \cdot \bar{e_2} + (1-\alpha) \cdot \underline{e_2} = \alpha \cdot d_2 + (1-\alpha) \cdot (E_{12} - d_1) = \alpha \cdot d_2 + (a-\alpha) \cdot E_{12} - (1-\alpha) \cdot d_1$$

Therefore, the fairness condition $e_1 - \tilde{e_1} = e_2 - \tilde{e_2}$ becomes:

$$e_1 - \alpha \cdot d_1 - (1-\alpha) \cdot E_{12} + (1-\alpha) \cdot d_2 = e_2 - \alpha \cdot d_2 - (1-\alpha) \cdot E_{12} + (1-\alpha) \cdot d_1$$

which simplifies to:

$$e_1 - \alpha \cdot d_1 + (1-\alpha) \cdot d_2 = e_2 - \alpha \cdot d_2 + (1-\alpha) \cdot d_1$$

and with a little algebra we get:

$$e_2 = E_{12} - \frac{E_{12} + d_1 - d_2}{2} = \frac{E_{12} - d_1 + d_2}{2}$$

which is, once more, the Talmudic distribution.

4 Conclusion

Based on Aumann's algorithmic solution of the "Talmudic Bankruptcy Code", Nash's definition of fairness, and Hurwicz's methodology for the fair cost of uncertainty, we have proven that the ancient code provides for a fair and equitable distribution of a bankruptcy estate.