The Financial Dynamics of Unsustainable Fraudulent Investment Schemes

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Abstract

We use a first order differential linear equation to describe the dynamics of an investment scheme that promises more than it can deliver (a Ponzi scheme). The model is based on (i) a promised, unrealistic, rate of return, (ii) the rate at which new investments are gathered, and (iii) the withdrawal rate. We establish the parameters that lead to the scheme's collapse.

1 Introduction

The principle of Ponzi schemes is simple: they entice potential investors by promising high returns, which they cannot possibly deliver. The only was to pay the promised returns is by attracting new investors whose money is used to subsidize the returns of those already in the fund.

In this article we posit a simple model that attempts answer the basic questions: How fast must new investments come in? How long can the scheme last? What parameters drive the dynamics of the scheme?

The model is described in Section 2, with detailed results on the behavior of the scheme as a function of seven parameters.

Section 3 highlights the main results and concludes.

2 The Model

2.1 Assumptions

We assume that the fund starts at time $t = 0$ with an initial investment $K \geq 0$, followed by a continuous cash inflow $s(t)$. We next assume a promised rate of return, s_p and a nominal rate r_n at which the money is actually invetsed. If $r_n \geq r_p$ then the fund is legitimate and has a profit rate $r_n - r_p$. If $r_n < r_p$ the fund is promising more than it can deliver. We call the promised rate, r_p, the "Ponzi rate".

We need to model the fact that investors withdraw at least part of their money along the way. The simplest way of doing this is to assume a constant withdrawal rate, r_w, applied at every time t to the promised accumulated capital. The withdrawal rate at time t by those who invested the initial amount K is $r_w K e^{t(r_p - r_w)}$. If r_w is less that the promised return r_p, then these withdrawals increase exponentially; r_w can also be larger than r_p, in which case withdrawals decrease exponentially, as these investors are eating into the capital K.

In order to calculate the withdrawals at time t from those who added to the fund between times 0 and t, we note that those who invested $s(u)$ at time u will want to withdraw a quantity $r_w s(u)e^{(r_p-r_w)(t-u)}$nat time $t > u$. Integrating these withdrawals from 0 to t and adding the previously calculated withdrawals from the initial deposit K, results in the following withdrawals at time t:

$$W(t) \overset{def.}{=} r_w \Big(K e^{t(r_p-r_w)} + \int_0^t s(u)e^{e(r_p-r_w)(t-u)} du \Big) \tag{1}$$

We note that the nominal interest rate, r_n, does not appear in $W(t)$: withdrawals are a function of only the promised rate of return, r_p.

2.2 The differential Equation

If S(t) is the amount in the fund at time t, then $S(t+dt)$ is obtained by adding to $S(t)$, the nominal return $r_n S(t)dt$, the inflow of fresh money $s(t)$, and subtracting the withdrawals $W(t)dt$:

$$S(t+dt) = S(t) + dt[r_n S_a(t) + s(t) - W(t)] \tag{2}$$

For $dt \to 0$, the amount $S(t)$ is the solution to the first-order linear differential equation:

$$\frac{S(t)}{dt} = r_n S_a(t) + s(t) - W(t) \tag{3}$$

We let $C = S(0)$ be the initial condition which may or may not be equal to K, the initial investment made by customers. The fund managers can make an initial "in-house" investment $K_0 \geq 0$, which will also be invested at the nominal rate r_n. In this case, the initial value $C = K_0 + K$ is greater than K. An initial condition $C < K$ formally corresponds to the cae where, for some reason (theft or other), a fraction of the initial investment K is not available. We will see later that the solution to the differential equation with an initial condition $C = S(0)$, other than K, will be used when there is, at some subsequent time t^*, a sudden change in parameter values (for example, if the cash inflow or withdrawal rate changes at t^*).

One often made assumption in the literature on the subject is that the cash inflow $s(t)$ grows at an exponential rate:

$$s(t) = s_0 e^{r_i t} \tag{4}$$

were s_0 is the initial density of the cash inflows and r_i will be called the investment growth rate. Then, Eq.(1) becomes:

$$W(t) = r_w e^{t(rp - r_w)} \Big(K + s_0 \frac{e^{t(r_w + r_t - r_p)} - 1}{r_w + r_i - r_p} \Big) \tag{5}$$

where the fraction should equal t when $r_w + r_i - r_p = 0$.

The solution, $S(t)$, to the differential equation (3) has a closed-form solution, which will be formulated using the function:

$$g(t, a, b, c, d, \alpha) \stackrel{def.}{=} ae^{bt} + ce^{dt} + \alpha \tag{6}$$

Based on this notation, $S(t)$ becomes:

$$S(t) = g(t, a, b, c, d, \alpha)e^{r_n t} = ae^{(b+r_n)t} + ce^{(d+r_n)t} + \alpha e^{r_n t} \tag{7}$$

where:

$$a \stackrel{def.}{=} \frac{r_w[s_0 - (r_i - r_p + r_w)K]}{(r_p - r_n - r_w)(r_i - r_p + r_w)}, \tag{8}$$

$$b \stackrel{def.}{=} r_p - r_n - r_w, \tag{9}$$

$$c \stackrel{def.}{=} \frac{s_0(r_i - r_p)}{(r_i - r_n)(r_i - r_p + r_w)}, \tag{10}$$

$$d \stackrel{def.}{=} r_i - r_n, \tag{11}$$

$$\alpha \stackrel{def.}{=} C - \frac{s_0(r_n - r_p) + Kr_w(r_i - r_n)}{(r_i - r_n)(r_n - r_p + r_w)}. \tag{12}$$

The solution $S(t)$ to Eq.(7) is a linear combination of three exponentials which we are not able to tackle directly by elementary methods. Instead, the zeros of $s(t)$ and of its derivative can be calculated numerically but providing no insights into the function's behavior.

On the other hand, if we know the number of positive zeros of $S(t)$ we can shed light on the conditions under which the fund is solvent ($S(t)$ remains positive). We will see below that, depending on parameter values, $S(t)$ of Eq. (7) has 0, 1, or 2 positive zeros. When there is no positive zero then $S(t)$ remains positive and the fuind is solvent. One positive zero means that $S(t)$ becomes negative and the scheme has collapsed. Two positive zeros mean that $S(t)$ becomes negative, reaches a negative minimum, then becomes positive again. The fund has collapsed but could recover with a bailout equal to the absolute value of the negative minimum.[1]. To simplify, we will say that, in this case, the scheme has collapsed and then recovered.

Analytical results on the number of positive zeros will result from noting that the zeros of $S(t)$ are also those of $g(t, a, b, c, d, \alpha)$ of Eq. (6). This function is a linear combination of only two exponentials plus a constant but the zeros still cannot be found in closed form by elementary method. However, the derivative of $g(t, a, b, c, d, \alpha)$ is a linear combination of two exponentials with no constant, which can be analytically studied. The following proposition provides results on the number of positive zeros of $g(t, a, b, c, d, \alpha)$.

Proposition 1. We consider the function $g(t, a, b, c, d, \alpha)$ of Eq. (6) in the non-trivial case where $a, b, c, d \neq 0$ and $b \neq d$. We also assume that $g(0, ab, c, d, \alpha) \geq 0$. We further consider the following set of two conditions:

$$U \overset{def.}{=} \frac{cd}{ab}, \quad V \overset{def.}{=} \frac{1 + cd/ab}{b - d} \leq 0 \tag{13}$$

The function $g(t, a, b, c, d, \alpha)$ then has an extremum:

$$m \overset{def.}{=} a\Big(\frac{-cd}{ab}\Big)^{\frac{b}{b-d}} + c\Big(\frac{-cd}{ab}\Big)^{\frac{d}{b-d}} + \alpha \tag{14}$$

at the positive value:

$$t_c \overset{def.}{=} \frac{ln\Big(\frac{-cd}{ab}\Big)}{b - d} \tag{15}$$

[1]See Bhattacharya (2003) for an economist's bailout model of a Ponzi scheme

if and only if Condition (13) is satisfied.

Depending on the number of positive zeros of $g(t, a, b, c, d, \alpha)$, we get results for four different cases A_1, A_2, A_3, A_4:

A_1: Condition (13) is satisfied and $ab + cd < 0$. If $m > 0$ then $g(t, a, b, c, d, \alpha)$ has no positive zero (and, therefore, remains positive). If $m < 0$ and $b, d, \alpha < 0$, then the function $g(t, a, b, c, d, \alpha)$ has exactly one positive zero at a value smaller than t_c. In all other cases, with $m < 0$, the function has one positive zero on each side of t_c.

A_2: Condition (13) is satisfied and $ab + cd > 0$. If $b, d < 0$, $\alpha > 0$, then the function has no positive zero. In all other cases there is one positive zero.

A_3: Condition (13) is not satisfied and $ab + cd < 0$. If $b, d < 0$, $\alpha > 0$, the function has no positive zero. In all other cases there is one positive zero.

A_4: Condition (13) is not satisfied and $ab + cd > 0$. There is no positive zero.

Proof. The proof is elementary and hinges on the following facts:

1. The derivative $g^{'}(t, a, b, c, d, \alpha)$ equals 0 at t_c and m is the value of $g(t, a, b, c, d, \alpha)$ at t_c; t_c is positive if and only if Condition (13) is satisfied. The derivative $g^{'}(0, a, b, c, d, \alpha)$ at 0 is equal to $ab + cd$.

2. If b and d are negative, the function $g(t, a, b, c, d, \alpha)$ tends to α for $t \to \infty$. If b or d is positive, the function tends to $\pm\infty$ (depending on the signs of a, c).

3. If Condition (13) is not satisfied, then either $g(t, a, b, c, d, \alpha)$ has an extremum for a negative value of t or no extremum at all. In both cases, the function $g(t, a, b, c, d, \alpha)$ for $t > 0$ is monotone increasing if $ab + cd > 0$ and monotone decreasing otherwise.

2.3 Main Result

In order to apply Proposition 1 to the parameters a, b, c, d, α of Eqs. (8)-(12), we first define:

$$\rho \overset{def.}{=} r_i - r_p, \ \sigma_K \overset{def.}{=} \frac{K r_w}{s_0} - 1. \tag{16}$$

We will need the function:

$$C_1(K) \overset{def.}{=} \frac{s_0(r_n - r_p) + K r_w (r_i - r_n)}{(r_i - r_n)(r_n - r_p + r_w)} \tag{17}$$

which is the critical value of C above which α of Eq. (12) is positive.

We then define the function:

$$Z(K) \overset{def.}{=} \frac{\frac{K}{s_0}(r_w + r_i - r_p) - 1}{(r_i - r_n)(r_n - r_p + r_w)}, \tag{18}$$

and note that $Z(K) = 1$ if and only if $k = s_0/r_w$ (i.e., $\sigma_K = 0$).

The extremum m of Eq. (14) and the corresponding t_c of Eq. (15) are:

$$m = s_0 \frac{(r_p - r_i) Z(K)^{\frac{r_n - r_i}{r_p - r_i - r_w}}}{(r_i - r_n)(r_n - r_p + r_w)} + C - C_1(K), \tag{19}$$

$$t_c = \frac{ln(Z(K))}{r_w + r_i - r_p}. \tag{20}$$

We also define the function $C_2(K)$ as:

$$C_2(K) \overset{def.}{=} C_1(K) + s_0 \frac{(r_p - r_i) Z(K)^{\frac{r_n - r_i}{r_p - R_i - r_w}}}{(r_i - r_n)(r_n - r_p + r_w)} \ if \ K \geq s_0/r_w, \tag{21}$$

$$C_2(K) \overset{def.}{=} 0 \ if \ K < s_0/r_w \tag{22}$$

The quantity $C_2(K)$ of Eq. (21) is the critical,value of C above which the extremum m of Eq. (19) is positive.

With these notations, we have the following result on the number of positive zeros of $S(t)$ of Eq. (7).

Theorem 1. We consider the solution $S(t)$ to Eq. (7) defined by the non-negative parameters K, C, s_0, r_i, r_w, r_p and r_n. The number of positive zeros of $S(t)$ is given as a function of the sign of ρ:

Case B_1: $\rho > 0$, $(r_i > r_p)$.

Sub-Case $B_{1,1}$: $\sigma_K < 0\,(K < s_0/r_w)$. $S(t)$ has no positive zero.

Sub-Case $B_{1,2}$: $\sigma_K > 0\,(K > s_0/r_w)$. We first consider the case $r_n > r_i$. If $C > C_2(K)$ (which includes the case $C = K$), then $S(t)$ has no positive zero for $t > 0$ and, therefore, remains positive for all $t > 0$. For $C_1(K) < C < C_2(K)$ the function $S(t)$ has one positive zero on each side of t_c. For $C < C_1(K)$ the function $S(t)$ has one positive zero. When $r_p < r_n < r_i$ the function $S(t)$ has one positive zero for $C < C_2(K)$ (which includes the case $C = K$) and none if $C > C_2(K)$. When $r_n < r_p$ the function $S(t)$ has one positive zero for $C < C_2(K)$ (which includes the case $C + K$ if K is larger than the fixed point $K^* = C_2(K^*)$ of $C_2(K)$) and none if $C > C_2(K)$ (which includes the case $C = K$ if K is smaller than the fixed point K^*).

Case B_2: $\rho < 0$, $(r_i < r_p)$.

Sub-Case $B_{2,1}$: $r_w < r_p - r_n$ or $r_n < r_i$. The function $S(t)$ has one positive zero.

Sub-Case $B_{2,2}$: $r_w > r_p - r_n$ and $r_n > r_i$. For $C > C_1(K)$ (which includes the case $C = K$ if $r_n > r_p$ the function $S(t)$ has no positive zero. For $C < C_1(K)$ (which includes the case $C = K$ if $r_n < r_p$), then $S(t)$ has one positive zero.

Proof. The application of Proposition 1 hinges on the following observations:

1. The parameter α of Eq. (12) is positive if and only if $C > C_1(K)$.

2. The extremum m of Eq. (14) is positive if and only if $C > C_2(K)$.

3. The difference $C_2(K) - C_1(K)$ has the same sign as $(r_p - r_i)/[(r_i - r_n)(r_n - r_p + r_w)]$.

4. For $\rho > 0$ the function $C_2(K)$ is a non-decreasing function of $K > 0$ that has no positive fixed point if $r_n > r_p$ and one positive fixed point $K^* = C_2(K^*)$ if $r_n < r_p$.

5. The parameter σ_K and the derivative

$$g^{'}(0,a,b,c,d,\alpha) = ab + cd = s_0 - Kr_w = -s_0\sigma_K \tag{23}$$

of $g(t,a,b,c,d,\alpha)$ at 0 have opposite signs.

With a,b,c,d,α of Eqs. (8)-(12) the quantities U and V of (13) are:

$$U = \frac{\rho}{-r_w\sigma_K - (\sigma_K+1)\rho}, \;\; V = \frac{\sigma_K}{-r_w\sigma_K - (\sigma_K+1)\rho}, \tag{24}$$

and are both negative if and only if ρ and σ_K have the same sign (because $\sigma_K + 1 > 0$).

In the case B_1 ($\rho > 0$) we considered two sub-cases:

Sub-Case $B_{1,1}$: $\sigma_K < 0$. Condition (13) is not satisfied and the derivative of $g(t,a,b,c,d,\alpha)$ at 0 is positive. This result follows from A_4 of Proposition 1.

Sub-Case $B_{1,2}$: $\sigma_K > 0$. Condition (13) is satisfied and the derivative of $g(t,a,b,c,d,\alpha)$ at 0 is positive. The sub-cases $B_{2,1}$ and$B_{2,2}$ correspond to $b = r_p - r_n - r_w$ or $d = r_i - r_n$ positive and to b and d negative, respectively. The results follow from A_2 of proposition 1. When $\sigma_K > 0$ Condition (13) is not satisfied and the derivative of $g(t,a,b,c,d,\alpha)$ at 0 is negative. These results follow from A_3 of Proposition 1.

2.4 Interpretation of Results

Theorem 1 breaks down the results depending on whether the rate of growth r_i of new investments is larger or smaller than the promised return r_p.

We first consider the case when r_i is larger than r_p (legitimate fund). In the case $B_{1,1}$ ($K < s_0/r_w$) the fund is solvent ($Z = 0$) regardless of the initial condition C. In the sub-case $B_{1,2}$ ($K > s_0/r_w$) the fund remains solvent when C remains above $C_2(K)$ (which includes the case $K = C$). For r_n between r_p and r_i the scheme collapses ($Z = 1$) as soon as c drops below $C_2(K)$. For R_n larger than r_i the fund collapses but recovers ($Z = 2$) if C does not fall too much below $C_2(K)$ ($C_1(K) < C < C_2(K)$). If C is too small($C < C_1(K)$) then the scheme collapses ($Z = 1$).

Case B_1 illustrates what happens to a Ponzi scheme $r_n < r_p$) even if the rate of growth of new investments r_i is larger than r_p. The fund will remain solvent for $C = K$ only if K is not too large (K less than the fixed point K^*). If $C = K$ and is larger than the fixed point K^*, then the combined withdrawals by the initial and subsequent investors eventually cause the scheme to collapse.

For Case B_2 (the rate of growth r_i of new investments is smaller than the promised return r_p) we first consider sub-case $B_{2,2}$ with $r_n > r_p$ (legitimate fund). The fund remains solvent for $C > C_1(K)$, which included the case $C = K$. In the Ponzi sub-case $B_{2,2}$, with $r_i < r_n < r_p$ and $r_w > rp_{rn}$ the fund does not grow too fast and is solvent if C is larger than $C_1(K)$, which is itself larger than K. This means that, despite an r_i and an r_n smaller than r_p, the Ponzi scheme is solvent if the manager can add to K an "in-house" investment K_0 at least equal to $C_1(K) - K$. This type of scheme is unprofitable of the manager and is often called a "philanthropic Ponzi scheme". This scenario relies on r_i remaining smaller than the nominal return r_n. If the manager does not invest enough ($C < C_1(K)$) the scheme collapses.

The Ponzi sub-case $B_{2,1}$ consists of the values $r_n < r_i$ and of the values (r_n, r_w) for which $r_w < r_p - r_n$ and r_n is between r_i and r_p. In this sub-case $B_{2,1}$ the fund grows too fast and collapses ($Z = 1$).

This analysis shows that the role of r_w is ambiguous when r_n is between r_i and r_p. Although a small r_w ($r_w < r_p - r_n$, $B_{2,1}$)may seem desirable, the fund will grow more in the long run and eventually collapses. A large r_w ($r_w > r_p - r_n$, $B_{2,2}$) may seem dangerous but depletes the fund and means smaller withdrawals in the long run. The fund is ultimately solvent if c is large enough to absorb the large early withdrawals ("philanthropic Ponzi scheme").

2.5 Actual and Promised Amounts in the Fund

In order to describe the dynamics of a fund that includes a suden parameter change at some time t^*, we need to make explicit the role of the parameters by denoting $S(t, K, C, s_0, r_i, r_w, r_p, r_n)$ the solution of differential equation (7).

We introduce the actual and promised amounts $S_a(t)$ and $S_p(t)$. The actual amounts $S_a(t)$ in the fund (resulting from the nominal rate of return r_n and the initial condition C), is the one given in Eq. (7) and rewritten explicitly as:

$$
\begin{aligned}
S_a(t) = S(t, K, C, s_0, r_i, r_w, r_p, r_n) = \frac{r_w[s_0 - (r_i - r_p + r_w)]K}{(r_p - r_n - r_w)(r_i - r_p + r_w)} e^{(r_p - r_w)t} \\
+ \frac{s_0(r_i - r_p)}{(r_i - r_n)(r_i - r_p + r_w)} e^{r_i t} \\
+ \Big(C - \frac{s_0(r_n - r_p) + K r_w (r_i - r_n)}{(r_i - r_n)(r_n - r_p + r_w)} \Big) e^{e r_n t}
\end{aligned}
$$

The value $S_p(t)$ is the amount that is promised to and belongs to investors; $S_p(t)$ is obtained by setting in Eq. (25) the parameter r_n equal to r_p and the initial condition C equal to K. Under these conditions, the third term in Eq. (25) becomes zero and:

$$
S_p(t) = S(t, K, K, s_0, r_i < r_w, r_p, r_p) = \frac{s_0}{r_p - r_i - r_w} (e^{(r_p - r_w)t} - e^{r_i t}) + K e^{(r_p - r_w)t} \quad (25)
$$

Contrary to the actual amount $S_a(t)$ in the fund, the promised amount $S_p(t)$ is positive regardless of the parameter values.

2.6 Change in Parameter Values

The assumption of an exponentially increasing density of new investments is not realistic in the long run and we may wish to examine what happens in the particular case where the irate of growth of new investments r_i suddenly drops to 0. This means that the flow of new investments becomes a constant. More generally, it would be useful to describe the future dynamics of the fund if at a point t^* the parameters$(s_0, r_i, r_w, r_p, r_n)$ experience a sudden (discontinuous) change of values and become $(s_0^{'}, r_i^{'}, r_w^{'}, r_p^{'}, r_n^{'})$.

We call $C^{'}$ and $K^{'}$ the actual and promised amounts in the fund at time t^*, respectively:

$$C^{'} = S_a(t^*) = S(t^*, K, C, s_0, r_i, r_w, r_p, r_n) \tag{26}$$

and

$$K^{'} = S_p(t^*) = S(t^*, K, K, s_0, r_i, r_w, r_p, r_p). \tag{27}$$

These values will be the initial condition and initial investment starting at time t^*. The actual and promised amounts at any time t then become:

$$S_a(t) = \begin{cases} S(t, K, C, s_0, r_i, r_w.r_p, r_n) & \text{if } t \leq t^*; \\ S(t - t^*, K^{'}, C^{'}, s_0^{'}, r_i^{'}, r_w^{'}, r_p^{'}, r_n^{'}) & \text{if } t > t^*. \end{cases} \tag{28}$$

and

$$S_p(t) = \begin{cases} S(t, K, K, s_0, r_i, r_w.r_p, r_p) & \text{if } t \leq t^*; \\ S(t - t^*, K^{'}, K^{'}, s_0^{'}, r_i^{'}, r_w^{'}, r_p^{'}, r_p^{'}) & \text{if } t > t^*. \end{cases} \tag{29}$$

Several discontinuous parameter changes at different times can be dealt with in this fashion.

3 Conclusion

As we expected, the fund is always solvent with $C = K$ in the case of a legitimate fund characterized by $r_n > r_p$. In a Ponzi scheme ($r_n < r_p$), the fund can remain solvent depending on the values of the rate of growth of new investments r_i and the withdrawal rate r_w. Our model sheds light on the ambiguous role played by these two parameters. If r_i is too large or r_w too small, the fund grows fast and can be in jeopardy as withdrawals increase. If r_i is too small or r_w too large, the fund may not keep up with withdrawals.

Our model yields a variety of increasing trajectories that may look alike initially, but are fundamentally different in their long-run behavior. Some will continue to increase as long as new investments come in – others will increase possibly for a long time before they collapse. This happens when parameter values in the phase spaces of solutions are close to border regions between different qualitative behaviors (for example, between no zero and one zero for the function $S(t)$). In some cases $S(t)$ initially decreases, reaches a positive or negative minimum, and then recovers.